A Null Test of the Cosmological Constant

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We provide a consistency relation between cosmological observables in general relativity with the cosmological constant. Breaking of this relation at any redshift would imply the breakdown of the hypothesis of the cosmological constant as an explanation of the current acceleration of the universe.

§1. Introduction

Recently, one of us provided an explicit relation between the luminosity distance and the growth rate of density perturbations which should hold under the assumptions that general relativity is the correct theory of gravity and that the dark energy clustering is negligible.[1)] In deriving the consistency relation, the Friedmann equation is not used, and hence the equation of state of dark energy, $w(z)$, is not assumed. $w(z)$ can be determined once the Friedmann equation (Einstein equation) is used.[2)–4)] Then, one may wonder whether another consistency relation, which is precisely zero for the cosmological constant, can be obtained if $w = -1$ is assumed, in such a way that the relation is precisely zero for the cosmological constant. The goal of this short note is to derive such a relation and with it to propose a null test for the cosmological constant. The basic idea is very simple: compare the matter density parameters determined from distance measurements assuming flat universe with the cosmological constant (Ω_{M0}^{w-1} hereafter) and those determined from other methods which are insensitive to the equation of state. We shall quantify the statement in the following.

§2. Equation of State of Dark Energy from Observations

First, from distance measurements, in terms of the coordinate distance $r(z) = d_L(z)/(1+z)$, the Hubble parameter is rewritten in a flat universe as[2)–4)]

$$H(z)^2 = \frac{1}{r'(z)^2}, \tag{1}$$

where the prime denotes the derivative with respect to z. Then, using the Friedmann equation for a flat universe,

$$H(z)^2 = H_0^2 \Omega_{M0}(1+z)^3 + \frac{8\pi G}{3}\rho_{\rm x}(z), \tag{2}$$

we obtain

$$8\pi G\rho_{\rm x}(z) = \frac{3}{r'(z)^2} - 3H_0^2\Omega_{M0}(1+z)^3, \tag{3}$$

where Ω_{M0} is the present matter density parameter and $\rho_{\rm x}(z)$ is the energy density of dark energy. From the time derivative of Eq. (3) and the energy-momentum conservation of dark energy, $\dot{\rho}_{\rm x} + 3H(1+w)\rho_{\rm x} = 0$, we obtain

$$(1+w(z))8\pi G\rho_{\rm x}(z) = -2\frac{(1+z)r''(z)}{r'(z)^3} - 3H_0^2\Omega_{M0}(1+z)^3. \tag{4}$$

Using Eqs.(3) and (4), $w(z)$ may be written as[2)–4)]

$$1 + w(z) = \frac{2(1+z)r''(z) + 3H_0^2\Omega_{M0}r'(z)^3(1+z)^3}{3r'(z)\left(H_0^2\Omega_{M0}r'(z)^2(1+z)^3 - 1\right)}. \tag{5}$$

Note that we do not assume any functional form of $w(z)$. At this stage, $w(z)$ determined from distance measurements exhibits degeneracy with Ω_{M0}. However, $\Omega_{M0}h^2$ can be determined from measurements of CMB anisotropies[5)] being insensitive to the equation of state. Moreover, from the evolution equation of $\delta(z)$ derived from the fluid equations,

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H_0^2\Omega_{M0}(1+z)^3\delta = 0, \tag{6}$$

Ω_{M0} can be written in terms of a density perturbation $\delta(z)$ independently of the equation of state (if dark energy is almost smooth) as[4)]

$$\Omega_{M0} = \frac{\delta'(0)^2}{3}\left(\int_0^\infty \frac{\delta(z)}{1+z}(-\delta'(z))dz\right)^{-1}. \tag{7}$$

§3. A Null Test of the Cosmological Constant

From Eq. (5), we define a function

$$\Omega_{M0}^{w=-1}(z) = -\frac{2r''(z)}{3H_0^2(1+z)^2r'(z)^3}, \tag{8}$$

which coincides with Ω_{M0} for the cosmological constant. Equating Eq. (8) with Eq. (7) thus gives a consistency relation for the cosmological constant,

$$\begin{aligned} \frac{\Omega_{M0}^{w=-1}(z)}{\Omega_{M0}} - 1 &= -\frac{2r''(z)}{3\Omega_{M0}H_0^2(1+z)^2r'(z)^3} - 1 \\ &= -\frac{2r''(z)}{H_0^2(1+z)^2r'(z)^3\delta'(0)^2}\int_0^\infty \frac{\delta(z)}{1+z}(-\delta'(z))dz - 1 = 0, \end{aligned} \tag{9}$$

which is the main result of this paper [*)] : compare $\Omega_{M0}^{w=-1}$ determined assuming a flat universe with the cosmological constant and Ω_{M0} determined from other methods

*) The time derivative of Eq. (8) for $w = -1$ gives the well-known relation of the jerk[6)] for the flat universe with/without the cosmological constant: $\frac{a^2(d^3a/dt^3)}{(da/dt)^3} = 1$. In this sense, Eq. (9) may be regarded as an integral form of the jerk relation.

which are insensitive to the equation of state. $r(z)$ can be determined from distant measurements of type Ia supernovae (SNIa),[7] gamma ray bursts (GRB)[8] and the baryon acoustic oscillation (BAO) in the matter power spectrum[9] and $\delta(z)$ can be determined from measurements of the weak lensing of galaxies[10] and of the evolution of cluster number density,[11] for example. If observational data indicate that the left-hand-side of Eq. (9) is nonzero at any redshift, this would be a clear signature of dynamical dark energy (or a non-flat universe). In this sense, Eq. (9) provides a null test of the cosmological constant.

In Fig. 1, the left-hand side of Eq. (9) is plotted for several values of $w(z)$: $w = -0.8, -0.9, -1, -1.1, -1.2$; $w(z) = -1(0)$ for $z \leq 2 (z > 2)$ *). Given $w(z)$, we compute $r(z)$ from Eq. (1) and δ from the evolution equation of δ. A flat universe with $\Omega_{M0} = 0.27$ is assumed. We note that Eq. (9) can be written in terms of $w(z)$ as

$$\frac{\Omega_{M0}^{w=-1}(z)}{\Omega_{M0}} - 1 = \frac{1-\Omega_{M0}}{\Omega_{M0}}(1+w(z))\exp\left(3\int_0^z \frac{w(z')}{1+z'}dz'\right). \tag{10}$$

If dark energy is not the cosmological constant, a deviation of more than 10% is expected at lower ($z < 1$) redshifts for a constant w. Moreover, even if dark energy behaves like the cosmological constant ($w = -1$) at lower redshifts, if it tracks matter ($w = 0$) at higher redshifts, which is the case in certain quintessence models with exponential potentials,[12] then a 10% deviation would be expected even at higher redshifts. Therefore, distance measurements at higher redshifts (by GRB and BAO) would be complementary to those at lower redshifts (by SNIa) in testing the cosmological constant.

Currently from the measurements of SNIa, assuming a flat universe with Λ, $\Omega_{M0}^{w=-1}$ is determined to be $\Omega_{M0}^{w=-1} = 0.29\pm^{0.05}_{0.03}$.[13]**) On the other hand, the measurements of CMB anisotropies by WMAP yields $\Omega_{M0}h^2 = 0.127 \pm 0.008$,[5] which when combined with the Hubble parameter determined by the HST,[14] $h = 0.72 \pm 0.08$, gives $\Omega_{M0} = 0.24 \pm 0.05$. Hence $\frac{\Omega_{M0}^{w=-1}}{\Omega_{M0}} - 1 = 0.21\pm^{0.46}_{0.38}$. The two Ω_{M0} coincides each other and the cosmological constant passes the null test. In any case, precision measurements of Ω_{M0} themselves can be a test of the cosmological constant.

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*) One may think that the constancy of $\Omega_{M0}^{w=-1}(z)$ determined from distance measurements alone for some redshift ranges could be a test of the cosmological constant. However, this second example shows explicitly that verifying $\Omega_{M0}^{w=-1} = \Omega_{M0}$ is crucial in testing the cosmological constant.

**) The same analysis of the gold dataset of Riess et al.(2007)[7] gives a slightly higher $\Omega_{M0}^{w=-1} = 0.34\pm^{0.05}_{0.03}$. Consequently, $\frac{\Omega_{M0}^{w=-1}}{\Omega_{M0}} - 1 = 0.42\pm^{0.50}_{0.42}$.

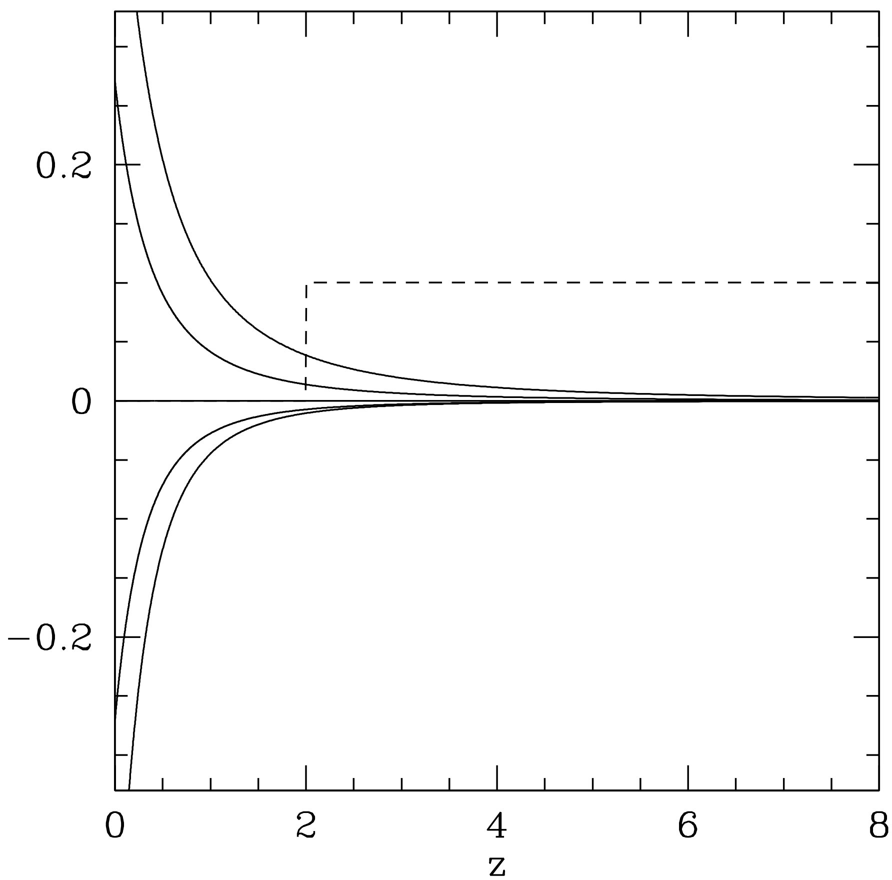

Fig. 1. Left-hand side of Eq.(9) as a function of z for several equation of state of dark energy. Solid lines are for $w = -0.8, -0.9, -1, -1.1, -1.2$ (from top to bottom) and the dashed line is for $w(z) = -1 (z \leq 2), 0 (z > 2)$.

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